PERGAMON

International Journal of Solids and Structures 40 (2003) 6839–6857

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Frictional interface crack in anisotropic bimaterial under combined shear and compression

J.C. Sung *, W.G. Chung

Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan

Received 8 December 2002

Abstract

The problem of a frictionally sliding interface crack embedded in an anisotropic bimaterial is investigated. Under remote normal compressive and shear load the crack faces are treated as completely closed in the direction of the normal load while in other directions the crack faces are allowed to slide. The frictional coefficient over the sliding zone is assumed to be constant. A set of singular integral equations is formulated which is valid for general anisotropic bimaterial. The nature of singularities for frictionally sliding bimaterial is investigated. It is found that for general anisotropic bimaterial the problem may be treated as a homogeneous anisotropy as long as $\tilde{\boldsymbol{W}} = \boldsymbol{0}$ and hence the stresses developed on the frictional surface would be uniform for such bimaterial. If $\tilde{\boldsymbol{W}}$ is not identically zero but with $(\tilde{\boldsymbol{W}})_{13} = 0$ and with the surface being frictionless then the stresses over the crack faces are square root singular. The homogeneous anisotropy satisfying $(\tilde{\boldsymbol{L}})_{12} = (\tilde{\boldsymbol{L}})_{32} = 0$ would transmit the load freely across the crack faces without any interference. For monoclinic bimaterial, the orders of singularities are found to depend on the material constant Λ and the frictional coefficient f and are either of order δ or $1 - \delta$ depending on the direction of the frictional force. The stresses on the interface for monoclinic bimaterial are also given explicitly.

Keywords: Anisotropic bimaterial; Interface crack; Friction

1. Introduction

Most of brittle materials such as ceramics, rocks, glasses and concrete etc. usually contain small and grain-sized faults that can be simulated as the problem of cracks embedded in an infinite inhomogeneous solid. When loaded in compression and shear, these cracks will propagate along the crack plane due to the effect of sliding frictional stresses and the propagation will continue leading eventually to the final failure of the structure. For some materials the sliding of the crack faces may even lead to the nucleation of tension cracks starting at the tips of the crack resulting in the so-called branched cracks. Above problems have been investigated by many researchers in the past two decades, but most of them are for isotropic materials. For instance Hoek et al. (1984), Horii and Nemat-Nasser (1982), Gorbatikh et al. (2001), Lauterbach and Gross

* Corresponding author. Tel.: +886-6-2757575-63133; fax: +886-6-2358542.
E-mail address: jcsung@mail.ncku.edu.tw (J.C. Sung).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2003.07.002

(1998), etc. As to isotropic bimaterial, a series of work have been done by Comninou and her co-workers (1977a,b, 1983). Recently a center frictional interfacial crack in an isotropic bimaterial based on the assumption of completely closed crack has been analyzed by Qian and Sun (1998). Their formulation is through the singular integral equations. Here we adopt the same approach but the results are valid for general anisotropic bimaterial.

In this paper, the problem of an interface crack embedded in an anisotrpic bimaterial is investigated. The bimaterial is subjected to compressive and shear loading at infinity. The crack surfaces are assumed to be able to slide in its own plane and the frictional coefficient is assumed to be constant over the sliding zone. The remote compressive force is applied so that crack faces perpendicular to the sliding direction are assumed to be completely closed. This problem is formulated in terms of a set of singular integral equations where the unknowns in the equations are the dislocation densities. It is found that for general anisotropic bimaterial the problem may be treated as a homogeneous anisotropy as long as $\tilde{\boldsymbol{W}} = \mathbf{0}$ and hence the stresses developed on the frictional surface would be uniform for such bimaterial. This is consistent with the near-tip analysis of the order of singularities for $\tilde{\boldsymbol{W}} = \mathbf{0}$ which is square root. If $\tilde{\boldsymbol{W}}$ is not identically zero, but with $\tilde{w}_2 = 0$ and with the surface being frictionless then the stresses developed on the crack faces are square root singular. The homogeneous anisotropy satisfying $(\tilde{\boldsymbol{L}})_{12} = (\tilde{\boldsymbol{L}})_{32} = 0$ would transmit the load freely across the crack faces without any interference. As to monoclinic bimaterial, the in-plane deformation is decoupled from the anti-plane part, however, the anti-plane part may have deformation due to the frictional force induced by the in-plane load. The orders of singularities at the crack tips, which depend on the material constant Λ and the frictional coefficient f, are either of order δ or $1-\delta$ depending on the direction of the frictional force. The dislocation densities and the stresses on the interface for monoclinic bimaterial are evaluated analytically and the results for isotropic materials may be recovered.

2. Basic equations

It is known that the displacement field $\mathbf{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ of a general anisotropic elastic material that undergoes a generalized plane strain deformation will satisfy, in the absence of body force, the following governing equation:

$$\mathbf{Q}\mathbf{u}_{,11} + (\mathbf{R} + \mathbf{R}^{\mathrm{T}})\mathbf{u}_{,12} + \mathbf{T}\mathbf{u}_{,22} = \mathbf{0} \tag{1}$$

where $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 matrices whose components are defined only by the material constants C_{ijks} as

$$\mathbf{Q} = [Q_{ik}] = [C_{i1k1}] \tag{2}$$

$$\mathbf{R} = [R_{ik}] = [C_{i1k2}] \tag{3}$$

$$\mathbf{T} = [T_{ik}] = [C_{i2k2}] \tag{4}$$

The general solutions of Eq. (1) may be expressed as (for more detailed information, please refer to Ting (1996)):

$$\mathbf{u} = \sum_{\alpha=1}^{3} \mathbf{a}_\alpha f_\alpha(z_\alpha) + \sum_{\alpha=1}^{3} \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha) \tag{5}$$

where f_α are arbitrary functions and $z_\alpha = x_1 + p_\alpha x_2$. $\mathbf{a}_\alpha$ $(\alpha = 1, 2, 3)$ and p_α are determined through the following eigenvalue problem

$$\{\mathbf{Q} + p_\alpha(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p_\alpha^2\mathbf{T}\}\mathbf{a}_\alpha = \mathbf{0} \quad (\text{no sum on } \alpha) \tag{6}$$

To evaluate the stresses, it would be convenient to construct the stress function as

$$\boldsymbol{\phi} = \sum_{\alpha=1}^{3} \mathbf{b}_\alpha f_\alpha(z_\alpha) + \sum_{\alpha=1}^{3} \bar{\mathbf{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha) \tag{7}$$

where $\mathbf{b}_\alpha$ $(\alpha = 1, 2, 3)$ are related to $\mathbf{a}_\alpha$ by

$$\mathbf{b}_\alpha = p_\alpha(\mathbf{R}^{\mathrm{T}} + p_\alpha \mathbf{T})\mathbf{a}_\alpha \quad \text{(no sum on } \alpha) \tag{8}$$

and the stresses τ_{1j} and τ_{2j} $(j = 1, 2, 3)$ may then be calculated by just differentiation of the stress function as

$$\tau_{1j} = -\phi_{j,2} \quad \text{and} \quad \tau_{2j} = \phi_{j,1} \tag{9}$$

In the present investigations, we are going to discuss the stress singularities at the tip of a crack with sliding surfaces. For such an investigation, it would be more convenient to express the general solutions of the displacement $\mathbf{u}$ (Eq. (5)) and the stress function ϕ (Eq. (7)) in the following form

$$\mathbf{u} = \mathbf{A}\langle f(z_*)\rangle\mathbf{q} + \mathbf{A}\langle f(\bar{z}_*)\rangle\tilde{\mathbf{q}} \tag{10}$$

$$\boldsymbol{\phi} = \mathbf{B}\langle f(z_*)\rangle\mathbf{q} + \mathbf{B}\langle f(\bar{z}_*)\rangle\tilde{\mathbf{q}} \tag{11}$$

where

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \tag{12}$$

$$\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3] \tag{13}$$

$$\langle f(z_*)\rangle = \begin{bmatrix} f(z_1) & 0 & 0 \\ 0 & f(z_2) & 0 \\ 0 & 0 & f(z_3) \end{bmatrix} \tag{14}$$

$$\langle f(\bar{z}_*)\rangle = \begin{bmatrix} f(\bar{z}_1) & 0 & 0 \\ 0 & f(\bar{z}_2) & 0 \\ 0 & 0 & f(\bar{z}_3) \end{bmatrix} \tag{15}$$

and $\mathbf{q}$ and $\tilde{\boldsymbol{q}}$ are arbitrary complex vectors.

3. Formulations of the problem

Let us consider an infinite bimaterial composed by two semi-infinite anisotropic materials. A crack with finite length $2c$ is situated on the interface which may slide over its crack faces. The bimaterial is subjected to shear stress S and normal compressive stress N at infinity. In the following analysis normal compressive stress N is assumed to be always negative. Due to the remote compressive loading the surface of the crack is assumed to be always closed in the direction of the compressive load, but is capable of sliding in its own plane. The frictional force fN may be developed on the crack faces if the surfaces are not frictionless. Here f is the frictional coefficient which is assumed to be constant over the crack faces in our analysis. Under these assumptions, the investigations of this frictional crack problem may be proceeded by the technique of modeling the crack by a continuous distribution of dislocations. Since the original problem is completely linear, we may separate the problem into a bimaterial problem that is free of any cracks with stresses $\mathbf{t}_2^\infty = (\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty)^{\mathrm{T}}$ applied at infinity and another problem that is induced by distributed dislocations $\mathbf{b}$ acting on the crack faces. The unknown densities of the distributed dislocations, which are to be sought, are chosen so that the boundary conditions of original problem are satisfied. To formulate this problem along this line, the fundamental solution due to a single concentrated dislocation acting on the crack-free bimaterial has to be constructed. This fundamental solution is introduced briefly in the following.

Based on Ting's solution (Ting, 1996), the fundamental solution for the displacement $\mathbf{u}$ and stress function ϕ due to a dislocation with Burger's vector $\mathbf{b}(=[b_1, b_2, b_3]^{\mathrm{T}})$ located on the interface of the bimaterial is

$$\mathbf{u}^{(1)} = \frac{-1}{\pi}(\ln r)\tilde{\mathbf{S}}\mathbf{b} - [\mathbf{S}^{(1)}(\theta)\tilde{\mathbf{S}} - \mathbf{H}^{(1)}(\theta)\tilde{\mathbf{L}}]\mathbf{b} \tag{16a}$$

$$\boldsymbol{\phi}^{(1)} = \frac{-1}{\pi}(\ln r)\tilde{\mathbf{L}}\mathbf{b} + [\mathbf{L}^{(1)}(\theta)\tilde{\mathbf{S}} + \mathbf{S}^{(1)^{\mathrm{T}}}(\theta)\tilde{\mathbf{L}}]\mathbf{b} \tag{16b}$$

for material 1, $x_2 > 0$ and

$$\mathbf{u}^{(2)} = \frac{-1}{\pi}(\ln r)\tilde{\mathbf{S}}\mathbf{b} - [\mathbf{S}^{(2)}(\theta)\tilde{\mathbf{S}} - \mathbf{H}^{(2)}(\theta)\tilde{\mathbf{L}}]\mathbf{b} \tag{17a}$$

$$\boldsymbol{\phi}^{(2)} = \frac{-1}{\pi}(\ln r)\tilde{\mathbf{L}}\mathbf{b} + [\mathbf{L}^{(2)}(\theta)\tilde{\mathbf{S}} + \mathbf{S}^{(2)^{\mathrm{T}}}(\theta)\tilde{\mathbf{L}}]\mathbf{b} \tag{17b}$$

for material 2, $x_2 < 0$, respectively. The matrices $\mathbf{S}^{(j)}(\theta)$, $\mathbf{H}^{(j)}(\theta)$ and $\mathbf{L}^{(j)}(\theta)$ $(j = 1, 2)$ in Eqs. (16a), (16b) and (17a), (17b) are periodic in θ with periodicity π. The matrices $\tilde{\mathbf{S}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{L}}$ of Eqs. (16a), (16b) and (17a), (17b) at the interface $(\theta = 0, \pi)$ are defined as

$$\tilde{\mathbf{S}} = -\tilde{\mathbf{H}}^{-1}(\mathbf{S}^{(1)^{\mathrm{T}}} + \mathbf{S}^{(2)^{\mathrm{T}}})(\mathbf{H}^{(1)} + \mathbf{H}^{(2)})^{-1} = -(\mathbf{L}^{(1)} + \mathbf{L}^{(2)})^{-1}(\mathbf{S}^{(1)^{\mathrm{T}}} + \mathbf{S}^{(2)^{\mathrm{T}}})\tilde{\mathbf{L}} \tag{18a}$$

$$\tilde{\mathbf{H}} = \{(\mathbf{L}^{(1)} + \mathbf{L}^{(2)}) + (\mathbf{S}^{(1)^{\mathrm{T}}} + \mathbf{S}^{(2)^{\mathrm{T}}})(\mathbf{H}^{(1)} + \mathbf{H}^{(2)})^{-1}(\mathbf{S}^{(1)} + \mathbf{S}^{(2)})\}^{-1} \tag{18b}$$

$$\tilde{\mathbf{L}} = \{(\mathbf{H}^{(1)} + \mathbf{H}^{(2)}) + (\mathbf{S}^{(1)} + \mathbf{S}^{(2)})(\mathbf{L}^{(1)} + \mathbf{L}^{(2)})^{-1}(\mathbf{S}^{(1)^{\mathrm{T}}} + \mathbf{S}^{(2)^{\mathrm{T}}})\}^{-1} \tag{18c}$$

which show that all these matrices are defined in terms of matrices $\mathbf{S}^{(j)}$, $\mathbf{H}^{(j)}$ and $\mathbf{L}^{(j)}$ $(j = 1, 2)$, known as Barnett–Lothe tensors. These Barnett–Lothe tensors are defined as

$$\mathbf{H}^{(j)} = 2\mathrm{i}\mathbf{A}^{(j)}\mathbf{A}^{(j)^{\mathrm{T}}}, \quad \mathbf{L}^{(j)} = -2\mathrm{i}\mathbf{B}^{(j)}\mathbf{B}^{(j)^{\mathrm{T}}} \quad \text{and} \quad \mathbf{S}^{(j)} = \mathrm{i}(2\mathbf{A}^{(j)}\mathbf{B}^{(j)^{\mathrm{T}}} - \mathbf{I}) \tag{19a}$$

which are all real and it can be shown that the following relations may be established

$$\mathbf{S}^{(j)} = \mathbf{S}^{(j)}(\pi), \quad \mathbf{H}^{(j)} = \mathbf{H}^{(j)}(\pi), \quad \mathbf{L}^{(j)} = \mathbf{L}^{(j)}(\pi). \tag{19b}$$

It is noted that matrices $\mathbf{H}^{(j)}$ and $\mathbf{L}^{(j)}$ possess the properties of symmetry and positive definite. When the bimaterial is made of identical material, then

$$\tilde{\mathbf{S}} = \frac{1}{2}\mathbf{S}^{(1)} = \frac{1}{2}\mathbf{S}^{(2)}, \quad \tilde{\mathbf{H}} = \frac{1}{2}\mathbf{H}^{(1)} = \frac{1}{2}\mathbf{H}^{(2)}, \quad \tilde{\mathbf{L}} = \frac{1}{2}\mathbf{L}^{(1)} = \frac{1}{2}\mathbf{L}^{(2)} \tag{19c}$$

Along the bimaterial interface, i.e., along x_1-axis, the stress function, for example, $\boldsymbol{\phi}^{(2)}$, may be written as

$$\boldsymbol{\phi}^{(2)} = \frac{1}{\pi}(\ln|x_1|)\tilde{\mathbf{L}}\mathbf{b} - H(-x_1)\tilde{\mathbf{W}}\mathbf{b} \tag{20}$$

where $H(x)$ is the Heaviside step function. It vanishes when $x < 0$ and has value 1 when $x > 0$. $\tilde{\mathbf{W}}$ in Eq. (20) is defined as $\tilde{\mathbf{W}} = \mathbf{L}^{(2)}\tilde{\mathbf{S}} + \mathbf{S}^{(2)^{\mathrm{T}}}\tilde{\mathbf{L}}$ which is skew-symmetric, the components of $\tilde{\mathbf{W}}$ are as follows

$$\tilde{\mathbf{W}} = \begin{bmatrix} 0 & \tilde{w}_3 & -\tilde{w}_2 \\ -\tilde{w}_3 & 0 & \tilde{w}_1 \\ \tilde{w}_2 & -\tilde{w}_1 & 0 \end{bmatrix} \tag{21}$$

With the known stress function due to a single dislocation applied on the interface, we may evaluate the traction on the interface as follows

$$\mathbf{t}_2(x_1, 0) = \frac{\partial \boldsymbol{\phi}^{(2)}(x_1, 0)}{\partial x_1} = \frac{1}{\pi x_1}\tilde{\mathbf{L}}\mathbf{b} + \delta(x_1)\tilde{\mathbf{W}}\mathbf{b} \tag{22}$$

where $\delta(x_1)$ is the delta function. Suppose now there are some dislocations with known densities distributed on the interface over a finite length, then the total tractions induced by these distributed dislocations may be obtained by just integrating the results of Eq. (22) with **b** replaced by the dislocation densities. Adding these tractions induced by these distributed dislocations to the tractions due to the applied stress at infinity, the total tractions $\mathbf{t}_2$ across the interface are therefore given as

$$\mathbf{t}_2(x_1, 0) = \mathbf{t}_2^\infty + \frac{-1}{\pi}\int_{-c}^{c} \frac{\tilde{\mathbf{L}}\mathbf{b}(\xi)}{\xi - x_1}\,\mathrm{d}\xi + \tilde{\mathbf{W}}\mathbf{b}(x_1) \tag{23}$$

where $\mathbf{b}(x_1)$ is the dislocation densities distributed over the frictional crack zone $(-c < x_1 < c)$ and $\mathbf{t}_2^\infty = (\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty)^{\mathrm{T}}$ is the stress applied at infinity. Since we are considering the problem of a crack capable of transmitting normal stress in x_2-direction which requires that the displacement jump vanish in that direction. Moreover, we consider the crack faces which are able to slide in other directions under combined compression and shear load. Therefore the following boundary conditions for the sliding crack faces, $|x_1| < c$, need to be satisfied

$$\mathbf{D}\mathbf{u}^{(1)} - \mathbf{D}\mathbf{u}^{(2)} = \mathbf{0} \tag{24a}$$

$$\mathbf{C}\mathbf{t}_2^{(1)} = \mathbf{0} = \mathbf{C}\mathbf{t}_2^{(2)} \tag{24b}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 0 \\ 0 & f & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24c}$$

and f is the frictional coefficient. Here we assume the frictional coefficients are the same in both x_1- and x_3-directions. Outside the crack zone, i.e., $|x_1| \geqslant c$, $x_2 = 0$, the displacements and tractions must be continuous

$$\mathbf{t}_2^{(1)} = \mathbf{t}_2^{(2)}, \tag{25a}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^{(2)} \tag{25b}$$

Hence applying the boundary conditions to Eq. (23), one obtains the set of singular integral equations for the dislocation densities **b** as

$$\mathbf{C}\mathbf{t}_2^\infty + \frac{-1}{\pi}\int_{-c}^{c} \frac{\mathbf{C}\tilde{\mathbf{L}}\mathbf{b}(\xi)}{\xi - x_1}\,\mathrm{d}\xi + \mathbf{C}\tilde{\mathbf{W}}\mathbf{b}(x_1) = \mathbf{0} \tag{26a}$$

Written explicitly, with $\mathbf{t}_2^\infty = (S, -N, 0)^{\mathrm{T}}$, the above equations become

$$\frac{-1}{\pi}\int_{-c}^{c} \begin{bmatrix} \tilde{L}_{11} + f\tilde{L}_{12} & \tilde{L}_{13} + f\tilde{L}_{23} \\ \tilde{L}_{13} + f\tilde{L}_{21} & \tilde{L}_{33} + f\tilde{L}_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \frac{\mathrm{d}\xi}{\xi - x_1} + \begin{bmatrix} -f\tilde{w}_3 & f\tilde{w}_1 - \tilde{w}_2 \\ \tilde{w}_2 - f\tilde{w}_3 & f\tilde{w}_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} -S + fN \\ fN \end{bmatrix} \tag{26b}$$

with $b_2(x_1) = 0$ due to the assumption that crack faces are closed in x_2-direction. To ensure unique solutions for the integral equations, the following conditions have to be satisfied

$$\int_{-c}^{c} b_1(\xi)\,\mathrm{d}\xi = 0 \quad \text{and} \quad \int_{-c}^{c} b_3(\xi)\,\mathrm{d}\xi = 0 \tag{27}$$

Note that Eq. (26b) are coupled equations for unknowns b_1 and b_3. If the crack faces are frictionless, the equations become

$$\frac{-1}{\pi}\int_{-c}^{c}\begin{bmatrix}\tilde{L}_{11} & \tilde{L}_{13}\\ \tilde{L}_{13} & \tilde{L}_{33}\end{bmatrix}\begin{bmatrix}b_1\\ b_3\end{bmatrix}\frac{d\xi}{\xi-x_1}+(-\tilde{w}_2)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}b_1\\ b_3\end{bmatrix}=\begin{bmatrix}-S\\ 0\end{bmatrix} \tag{28a}$$

These equations are still coupled. However if this frictionless bimaterial have further properties such that $\tilde{w}_2 = 0$ then the solutions of these equations can be treated easily and the behavior of the dislocation densities will possess the well known square root singularity. If the bimaterial are such that the following condition

$$\tilde{w}_2 = f(\tilde{w}_1 + \tilde{w}_3) \tag{28b}$$

is satisfied, then Eq. (26b) becomes

$$\frac{-1}{\pi}\int_{-c}^{c}\begin{bmatrix}\tilde{L}_{11}+f\tilde{L}_{12} & \tilde{L}_{13}+f\tilde{L}_{23}\\ \tilde{L}_{13}+f\tilde{L}_{21} & \tilde{L}_{33}+f\tilde{L}_{23}\end{bmatrix}\begin{bmatrix}b_1\\ b_3\end{bmatrix}\frac{d\xi}{\xi-x_1}+f\begin{bmatrix}-\tilde{w}_3 & -\tilde{w}_3\\ \tilde{w}_1 & \tilde{w}_1\end{bmatrix}\begin{bmatrix}b_1\\ b_3\end{bmatrix}=\begin{bmatrix}-S+fN\\ fN\end{bmatrix} \tag{28c}$$

Suppose the solid is a homogeneous medium, then equations become

$$\frac{-1}{2\pi}\int_{-c}^{c}\begin{bmatrix}L_{11}+fL_{12} & L_{13}+fL_{23}\\ L_{13}+fL_{21} & L_{33}+fL_{23}\end{bmatrix}\begin{bmatrix}b_1\\ b_3\end{bmatrix}\frac{d\xi}{\xi-x_1}=\begin{bmatrix}-S+fN\\ fN\end{bmatrix} \tag{28d}$$

since $\tilde{\mathbf{W}} = \mathbf{0}$ and $2\tilde{\mathbf{L}} = \mathbf{L}$ for homogeneous solids.

4. Stress singularities at a closed frictionally interfacial crack

Before solving the singular integral equations for the dislocation densities derived in the previous section, we would first discuss the stress singularities at the interface crack tip. Although the nature of singularities at the frictionally closed crack tips for general anisotropic bimaterial may be analyzed directly from the singular equations, it would be more straightforward to adopt the common approach of near-tip expansion method. To do that, we may choose the complex function $\langle f(z_*)\rangle$ in Eq. (10) to be in the form

$$\langle f(z_*)\rangle = \langle z_*^{\delta}\rangle$$

therefore the displacement field $\mathbf{u}$ and stress function $\boldsymbol{\phi}$ may be expressed as

$$\mathbf{u} = r^{\delta+1}\{\mathbf{A} < \zeta_*^{\delta+1}(\theta) > \mathbf{q} + \bar{\mathbf{A}} < \bar{\zeta}_*^{\delta+1}(\theta) > \tilde{\mathbf{q}}\} \tag{29a}$$

$$\boldsymbol{\phi} = r^{\delta+1}\{\mathbf{B} < \zeta_*^{\delta+1}(\theta) > \mathbf{q} + \bar{\mathbf{B}} < \bar{\zeta}_*^{\delta+1}(\theta) > \tilde{\mathbf{q}}\} \tag{29b}$$

where

$$\zeta_*(\theta) = \cos(\theta) + p_*\sin(\theta) \tag{30}$$

Now, consider the bimaterial that consists of two dissimilar anisotropic half-planes. Let material 1 occupies the upper half-plane

$$r > 0,\quad 0 \leqslant \theta \leqslant \pi$$

while the lower half-plane

$$r > 0,\quad -\pi \leqslant \theta \leqslant 0$$

is occupied by material 2. Employing Eqs. (29a) and (29b) for materials 1 and 2, we may have

$$\mathbf{u}^{(i)}(r,\theta) = r^{\delta+1}\left\{\mathbf{A}^{(i)} < \zeta_*^{\delta+1}(\theta) > \mathbf{q}^{(i)} + \bar{\mathbf{A}}^{(i)} < \bar{\zeta}_*^{\delta+1}(\theta) > \tilde{\mathbf{q}}^{(i)}\right\} \tag{31a}$$

$$\boldsymbol{\phi}^{(i)}(r,\theta) = r^{\delta+1}\left\{\mathbf{B}^{(i)} < \zeta_*^{\delta+1}(\theta) > \mathbf{q}^{(i)} + \bar{\mathbf{B}}^{(i)} < \bar{\zeta}_*^{\delta+1}(\theta) > \tilde{\mathbf{q}}^{(i)}\right\} \tag{31b}$$

where the superscript (i) $(i = 1, 2)$ represents quantities related to material 1 or 2, respectively. It is noted that on the surface $\theta = 0$ and $\theta = \pm\pi$, variable defined in (30) takes the following special values

$$\zeta_*^{(i)^{\delta+1}}(0) = 1, \quad \zeta_*^{(i)^{\delta+1}}(\pi) = \mathrm{e}^{\mathrm{i}\delta\pi}, \quad \zeta_*^{(i)^{\delta+1}}(-\pi) = -\mathrm{e}^{-\mathrm{i}\delta\pi} \tag{32}$$

Since the two anisotropic materials are perfectly bonded along the interface $\theta = 0$ while the surfaces at $\theta = \pm\pi$ are capable of sliding in both x_1- and x_3-directions and are always closed in x_2-direction, hence the continuities of displacements and tractions at the interface $\theta = 0$ require

$$u_j^{(1)}(r,0) = u_j^{(2)}(r,0), \quad \phi_j^{(1)}(r,0) = \phi_j^{(2)}(r,0), \quad (j = 1,2,3) \tag{33}$$

while the sliding boundary conditions at $\theta = \pm\pi$ may be described as

$$u_2^{(1)}(r,\pi) = u_2^{(2)}(r,-\pi), \tag{34a}$$

$$\sigma_{21}^{(2)}(r,-\pi) = -f\sigma_{22}^{(2)}(r,-\pi), \quad \sigma_{23}^{(2)}(r,-\pi) = -f\sigma_{22}^{(2)}(r,-\pi) \tag{34b}$$

where f denotes the Coulomb's friction coefficient. Eqs. (34a) and (34b) may be expressed in more compact form as

$$\mathbf{D}\mathbf{u}^{(1)}(r,\pi) = \mathbf{D}\mathbf{u}^{(2)}(r,-\pi), \quad \mathbf{C}\mathbf{t}^{(1)}(r,\pi) = \mathbf{0} = \mathbf{C}\mathbf{t}^{(2)}(r,-\pi) \tag{34c}$$

where **C** and **D** are defined in Eq. (24c). Using the conditions that the surface $\theta = 0$ should be perfectly bonded, Eqs. (31a) and (31b) will lead to the following results

$$\mathbf{A}^{(1)}\mathbf{q}^{(1)} + \bar{\mathbf{A}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{A}^{(2)}\mathbf{q}^{(2)} + \bar{\mathbf{A}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{35a}$$

$$\mathbf{B}^{(1)}\mathbf{q}^{(1)} + \bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{B}^{(2)}\mathbf{q}^{(2)} + \bar{\mathbf{B}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{35b}$$

In order to apply the conditions at $\theta = \pm\pi$, we need the displacement and stress function at the surface $\theta = \pm\pi$ to satisfy

$$\mathbf{u}^{(i)}(r,\pm\pi) = -r^{\delta+1}\left\{\mathrm{e}^{\pm\mathrm{i}\delta\pi}\mathbf{A}^{(i)}\mathbf{q}^{(i)} + \mathrm{e}^{\mp\mathrm{i}\delta\pi}\bar{\mathbf{A}}^{(i)}\tilde{\mathbf{q}}^{(i)}\right\} \tag{36a}$$

$$\boldsymbol{\phi}^{(i)}(r,\pm\pi) = -r^{\delta+1}\{\mathrm{e}^{\pm\mathrm{i}\delta\pi}\mathbf{B}^{(i)}\mathbf{q}^{(i)} + \mathrm{e}^{\mp\mathrm{i}\delta\pi}\bar{\mathbf{B}}^{(i)}\tilde{\mathbf{q}}^{(i)}\} \tag{36b}$$

Since the tractions are perfectly continuous along the sliding surface $\theta = \pm\pi$, i.e.,

$$\phi_i^{(1)}(r,\pi) = \phi_i^{(2)}(r,-\pi), \quad (i = 1,2,3) \tag{37a}$$

use of above condition in Eq. (36b) would get

$$\mathrm{e}^{\mathrm{i}\delta\pi}\mathbf{B}^{(1)}\mathbf{q}^{(1)} + \mathrm{e}^{-\mathrm{i}\delta\pi}\bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathrm{e}^{-\mathrm{i}\delta\pi}\mathbf{B}^{(2)}\mathbf{q}^{(2)} + \mathrm{e}^{\mathrm{i}\delta\pi}\bar{\mathbf{B}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{37b}$$

or

$$\mathrm{e}^{2\mathrm{i}\delta\pi}\mathbf{B}^{(1)}\mathbf{q}^{(1)} + \bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{B}^{(2)}\mathbf{q}^{(2)} + \mathrm{e}^{2\mathrm{i}\delta\pi}\bar{\mathbf{B}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{37c}$$

Comparing (37c) and (35b), one finds that

$$\mathbf{q}^{(2)} = (\mathbf{B}^{(2)})^{-1}\bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} \tag{38a}$$

$$\tilde{\mathbf{q}}^{(2)} = (\bar{\mathbf{B}}^{(2)})^{-1}\mathbf{B}^{(1)}\mathbf{q}^{(1)} \tag{38b}$$

Now considering the boundary conditions at $\theta = \pm\pi$ which are expressed in Eq. (34c). Enforcing these conditions will lead to

$$e^{i\delta\pi}\mathbf{D}\mathbf{A}^{(1)}\mathbf{q}^{(1)} + e^{-i\delta\pi}\mathbf{D}\bar{\mathbf{A}}^{(1)}\tilde{\mathbf{q}}^{(1)} = e^{-i\delta\pi}\mathbf{D}\mathbf{A}^{(2)}\mathbf{q}^{(2)} + e^{i\delta\pi}\mathbf{D}\bar{\mathbf{A}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{39a}$$

$$e^{i\delta\pi}\mathbf{C}\mathbf{B}^{(1)}\mathbf{q}^{(1)} + e^{-i\delta\pi}\mathbf{C}\bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{0} \tag{39b}$$

Adding these two equations, one obtains

$$e^{i\delta\pi}(\mathbf{C}\mathbf{B}^{(1)} + \mathbf{D}\mathbf{A}^{(1)})\mathbf{q}^{(1)} + e^{-i\delta\pi}(\mathbf{C}\bar{\mathbf{B}}^{(1)} + \mathbf{D}\bar{\mathbf{A}}^{(1)})\tilde{\mathbf{q}}^{(1)} = e^{-i\delta\pi}\mathbf{D}\mathbf{A}^{(2)}\mathbf{q}^{(2)} + e^{i\delta\pi}\mathbf{D}\bar{\mathbf{A}}^{(2)}\tilde{\mathbf{q}}^{(2)} \tag{40}$$

Substituting Eqs. (38a) and (38b) into (35a) and (40), one obtains

$$\mathbf{U}\mathbf{B}^{(1)}\mathbf{q}^{(1)} + \overline{\mathbf{U}\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{0} \tag{41}$$

$$e^{2i\delta\pi}(\mathbf{C} + \mathbf{D}\mathbf{U})\mathbf{B}^{(1)}\mathbf{q}^{(1)} + (\mathbf{C} - \mathbf{D}\bar{\mathbf{U}})\bar{\mathbf{B}}^{(1)}\tilde{\mathbf{q}}^{(1)} = \mathbf{0} \tag{42}$$

where

$$-i\mathbf{U} = \mathbf{M}^{(1)^{-1}} + \bar{\mathbf{M}}^{(2)^{-1}} = \hat{\mathbf{D}} - i\hat{\mathbf{W}} \tag{43}$$

and

$$\mathbf{M}^{(j)} = -i\mathbf{B}^{(j)}\mathbf{A}^{(j)^{-1}}, \quad j = 1,2 \tag{44}$$

$$\hat{\mathbf{W}} = \begin{bmatrix} 0 & \hat{w}_3 & -\hat{w}_2 \\ -\hat{w}_3 & 0 & \hat{w}_1 \\ \hat{w}_2 & -\hat{w}_1 & 0 \end{bmatrix} = \mathbf{S}^{(1)}(\mathbf{L}^{(1)})^{-1} - \mathbf{S}^{(2)}(\mathbf{L}^{(2)})^{-1} \tag{45}$$

$$\hat{\mathbf{D}} = \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} & \hat{D}_{13} \\ \hat{D}_{12} & \hat{D}_{22} & \hat{D}_{23} \\ \hat{D}_{13} & \hat{D}_{23} & \hat{D}_{33} \end{bmatrix} = (\mathbf{L}^{(1)})^{-1} + (\mathbf{L}^{(2)})^{-1} \tag{46}$$

Eliminating $\tilde{\mathbf{q}}^{(2)}$ from Eqs. (41) and (42) leads to an equation for $\mathbf{q}^{(1)} = \mathbf{0}$ and from the nontrivial solution of $\mathbf{q}^{(1)}$ the characteristic equation for the order of stress singularities may be obtained as

$$\|\mathbf{E}(\delta)\| = \|\mathbf{C}(e^{2i\delta\pi}\mathbf{U}^{-1} - \bar{\mathbf{U}}^{-1}) + (e^{2i\delta\pi} - 1)\mathbf{D}\| = 0 \tag{47}$$

Since the inverse of the matrix $\mathbf{U}$ may be expressed as

$$\mathbf{U}^{-1} = -i(\mathbf{M}^{(1)^{-1}} + \bar{\mathbf{M}}^{(2)^{-1}})^{-1} = \tilde{\mathbf{W}} - i\tilde{\mathbf{L}} \tag{48}$$

where $\tilde{\mathbf{L}}$ is symmetric and positive definite, and $\tilde{\mathbf{W}}$ is skew-symmetric, both are 3×3 real matrices which may be expressed as

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} \\ \tilde{L}_{12} & \tilde{L}_{22} & \tilde{L}_{23} \\ \tilde{L}_{13} & \tilde{L}_{23} & \tilde{L}_{33} \end{bmatrix} = (\hat{\mathbf{D}} - \hat{\mathbf{W}}^{\mathrm{T}}\hat{\mathbf{D}}^{-1}\hat{\mathbf{W}})^{-1}, \tag{49a}$$

$$\tilde{\mathbf{W}} = \begin{bmatrix} 0 & \tilde{w}_3 & -\tilde{w}_2 \\ -\tilde{w}_3 & 0 & \tilde{w}_1 \\ \tilde{w}_2 & -\tilde{w}_1 & 0 \end{bmatrix} = \hat{\mathbf{D}}^{-1}\hat{\mathbf{W}}\tilde{\mathbf{L}} = \tilde{\mathbf{L}}\hat{\mathbf{W}}\hat{\mathbf{D}}^{-1} \tag{49b}$$

the characteristic equation may be further rewritten as

$$\|\mathbf{E}(\delta)\| = \|(\mathrm{e}^{2\mathrm{i}\delta\pi} - 1)(\mathbf{C}\tilde{\mathbf{W}} + \mathrm{i}\lambda\mathbf{C}\tilde{\mathbf{L}} + \mathbf{D})\| = 0 \quad (50a)$$

where

$$\lambda = \frac{1 + \mathrm{e}^{2\mathrm{i}\delta\pi}}{1 - \mathrm{e}^{2\mathrm{i}\delta\pi}} = \mathrm{i}\cot\delta\pi, \quad \mathrm{e}^{2\mathrm{i}\delta\pi} = -\frac{1-\lambda}{1+\lambda} \quad (50b)$$

The expansion of Eq. (50a) is

$$(\mathrm{e}^{2\mathrm{i}\delta\pi} - 1)^3[a(\cot\delta\pi)^2 - b\cot\delta\pi + c] = 0 \quad (51a)$$

where

$$a = (\tilde{L}_{11}\tilde{L}_{33} - \tilde{L}_{13}^2) + f(\tilde{L}_{12}\tilde{L}_{33} + \tilde{L}_{11}\tilde{L}_{23} - \tilde{L}_{13}\tilde{L}_{23} - \tilde{L}_{13}\tilde{L}_{12}) \quad (51b)$$

$$b = f[\tilde{w}_1(\tilde{L}_{11} - \tilde{L}_{13}) + \tilde{w}_2(\tilde{L}_{12} - \tilde{L}_{23}) + \tilde{w}_3(\tilde{L}_{13} - \tilde{L}_{33})] \quad (51c)$$

$$c = \tilde{w}_2^2 - f\tilde{w}_2(\tilde{w}_1 + \tilde{w}_3) \quad (51d)$$

From Eq. (51a) we find that the characteristic equation for δ is

$$a(\cot\delta\pi)^2 - b\cot\delta\pi + c = 0 \quad (52a)$$

since $(\mathrm{e}^{2\mathrm{i}\delta\pi} - 1) \neq 0$. This is the characteristic equation for the frictional cracks for general anisotropic bimaterial. It is known that for isotropic bimaterial the root is −1/2 for frictionless sliding crack problem and the roots will alter but remain real for the frictional case (Comninou (1977b)). For the present anisotropic bimaterial, the roots of Eq. (52a) may be real or complex depending on the values of a, b and c. For example, for the frictionless crack faces Eq. (52a) becomes

$$a\cot^2\delta\pi + c = 0 \quad (52b)$$

with $a = \tilde{L}_{11}\tilde{L}_{33} - \tilde{L}_{13}^2$ and $c = \tilde{w}_2^2$. Therefore the roots for the frictionless problem are

$$\delta = \cot^{-1}\left(\pm\mathrm{i}\tilde{w}_2/\sqrt{\tilde{L}_{11}\tilde{L}_{33} - \tilde{L}_{13}^2}\right)/\pi \quad (52c)$$

which are usually complex. In the following we will discuss the roots for some special materials. Let's first consider the homogeneous anisotropic materials. Since $\tilde{\mathbf{W}} = \mathbf{0}$ and $2\tilde{\mathbf{L}} = \mathbf{L}$ for homogeneous material, it is seen that Eqs. (51b)–(51d) become

$$a = \frac{1}{4}[(L_{11}L_{33} - L_{13}^2) + f(L_{12}L_{33} + L_{11}L_{23} - L_{13}L_{23} - L_{13}L_{12})] \quad (53a)$$

$$b = 0, \quad c = 0 \quad (53b)$$

Therefore the characteristic equation becomes

$$(\mathrm{e}^{2\mathrm{i}\delta\pi} - 1)^3 a(\cot\delta\pi)^2 = 0 \quad \text{or} \quad \cot\delta\pi = 0, \quad (53c)$$

so that $\delta = -1/2$. Hence for a homogeneous anisotropic solid the stress singularity for a closed crack is square root no matter whether the frictional resistance exists or not.

4.1. Anisotropic bimaterial

Next let's consider the anisotropic bimaterial having the property of $\tilde{\mathbf{W}} = \mathbf{0}$, then we immediately find, from Eq. (51), that $b = c = 0$ for this material so that the near tip behavior would be the same as the homogeneous material as discussed above. Suppose $\tilde{\mathbf{W}}$ is not identically zero, but that $(\tilde{\mathbf{W}})_{13} = -\tilde{w}_2 = 0$ and $f = 0$ then apparently $b = c = 0$ again, hence the square root singular behavior near the tip will occur for such frictionless anisotropic bimaterial. If the anisotropic bimaterial are such that the parameter c defined by Eq. (51d) is zero which results in

$$\tilde{w}_2^2 = f\tilde{w}_2(\tilde{w}_1 + \tilde{w}_3) \tag{54a}$$

then obviously from Eq. (51a) the characteristic roots for such materials are

$$\delta = -1/2 \quad \text{and} \quad \delta = \cot^{-1}(b/a)/\pi \tag{54b}$$

which shows that the roots are all real even if the crack faces possess frictions. Note that the conditions specified by Eq. (54a) have two possibilities, one is $\tilde{w}_2 = 0$ and the other is $\tilde{w}_2 = f(\tilde{w}_1 + \tilde{w}_3)$. Both cases are valid even the frictional coefficient is not zero.

4.2. Monoclinic bimaterial

Next consider monoclinic bimaterial with symmetry plane at $x_3 = 0$. Applying $\tilde{w}_1 = \tilde{w}_2 = 0$, $\tilde{L}_{13} = \tilde{L}_{23} = 0$ for monoclinic bimaterial, the corresponding characteristic equation (Eq. (51a)) become

$$(e^{2i\delta\pi} - 1)^3[a(\cot\delta\pi)^2 - b\cot\delta\pi] = 0 \tag{55a}$$

where

$$a = (\tilde{L}_{11} + f\tilde{L}_{12})\tilde{L}_{33} \tag{55b}$$

$$b = -f\tilde{w}_3\tilde{L}_{33}, \quad c = 0 \tag{55c}$$

Apparently one of the root of Eq. (55a), which is −1/2, corresponds to the anti-plane problem while the rest corresponding to in-plane problem may be found from the following

$$\cot\delta\pi = -f\Lambda \tag{55d}$$

where

$$\Lambda = \frac{\tilde{w}_3}{\tilde{L}_{11} + f\tilde{L}_{12}} \tag{55e}$$

The results for in-plane mode are then

$$\delta = -\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(f\Lambda), \quad -\pi/2 < \tan^{-1}(f\Lambda) < \pi/2 \tag{56}$$

which shows that the singularity order is always real, no complex roots exist. These real roots are dependent on the friction coefficient and on the material constants only through the parameter Λ defined by Eq. (55e). The order of singularity is usually not square root, but it would become square root for the frictionless case. The order of singularity versus frictional coefficient is plotted in Fig. 1 for several values of Λ. Note that the material parameter Λ entering in the characteristic equation may also be expressed in terms of reduced elastic compliance. The result is

$$\Lambda = \frac{w^{(1)} - w^{(2)}}{s_{11}'^{(1)}(e'^{(1)} - fd'^{(1)}) + s_{11}'^{(2)}(e'(2) - fd'^{(2)})} \tag{57}$$

where $s_{mn}'^{(i)}$ $(i = 1, 2)$ are the reduced elastic compliance defined in Eq. (A.5) (see Appendix A) and those constants $e'^{(i)}$, $g'^{(i)}$ and $d'^{(i)}$ are defined in Eq. (A.3). If the monoclinic bimaterial are degenerated to orthotropic bimaterial, we find that the parameter Λ becomes

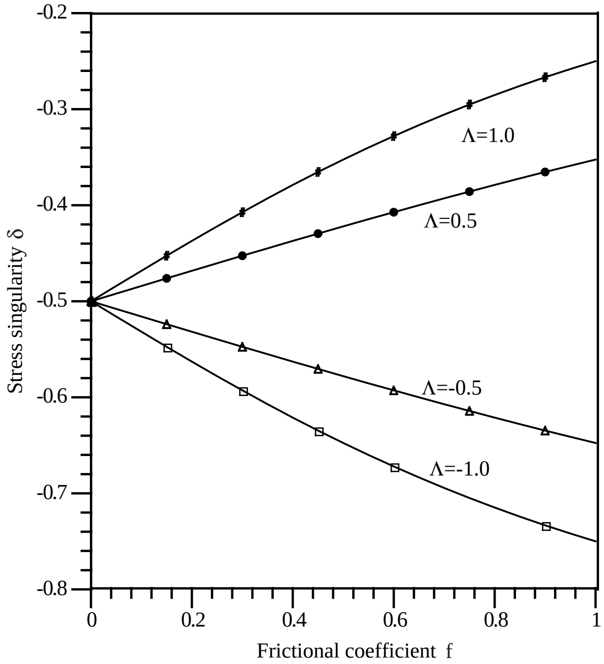

Fig. 1. The variations of the stress singularity δ versus frictional coefficient f for several material constant Λ.

$$\Lambda = \frac{\tilde{w}_3}{\tilde{L}_{11}} \tag{58}$$

since $a = \tilde{L}_{11}\tilde{L}_{33}$, $b = -f\tilde{w}_3\tilde{L}_{33}$ and $c = 0$ for orthotropic bimaterial. Expressed in terms of elastic compliance, Eq. (58) becomes

$$\Lambda = \frac{w^{(1)} - w^{(2)}}{s_{11}'^{(1)}e'^{(1)} + s_{11}'^{(2)}e'^{(2)}} \tag{59}$$

Defining for orthotropic bimaterial the generalized Dundurs's constants $\hat{\alpha}$ and $\hat{\beta}$ as (Poonsawat et al., 2001; Ting, 1996)

$$\hat{\alpha} = \frac{s_{11}'^{(1)}e'^{(1)} - s_{11}'^{(2)}e'^{(2)}}{s_{11}'^{(1)}e'^{(1)} + s_{11}'^{(2)}e'^{(2)}}, \quad \hat{\beta} = \frac{w^{(1)} - w^{(2)}}{s_{11}'^{(1)}e'^{(1)} + s_{11}'^{(2)}e'^{(2)}} \tag{60}$$

one immediately see that parameter Λ defined above is exactly the same as $\hat{\beta}$. Hence we may conclude that the stress singularities for orthotropic bimaterial will depend on the generalized Dundurs constant $\hat{\beta}$ only, it is totally independent of the other constant $\hat{\alpha}$.

4.3. Isotropic bimaterial

If the anisotropic bimaterial are further degenerated to isotropic bimaterial, we find that for isotropic bimaterial

$$\hat{\mathbf{W}} = \hat{k}_1\mathbf{J} \quad \hat{\mathbf{D}} = \hat{k}_2\mathbf{I}_1 + \hat{k}_3\mathbf{I}_2 \tag{61a}$$

where

$$\hat{k}_1 = \frac{\mu_2(1-2\nu_1) - \mu_1(1-2\nu_2)}{2\mu_1\mu_2}, \quad \hat{k}_2 = \frac{\mu_2(1-\nu_1) + \mu_1(1-\nu_2)}{\mu_1\mu_2}, \quad \hat{k}_3 = \frac{\mu_2 + \mu_1}{\mu_1\mu_2} \tag{61b}$$

and the constants a, b and c are

$$a = \frac{\hat{k}_2^3}{\hat{k}_3(\hat{k}_2^2 + \hat{k}_1)^2}, \quad b = -f\tilde{w}_3\frac{\hat{k}_2^2}{\hat{k}_3(\hat{k}_2^2 + \hat{k}_1)}, \quad c = 0, \tag{61c}$$

where $\tilde{w}_3 = -\hat{k}_1/(\hat{k}_2^2 + \hat{k}_1)$. With these results, we find that $\Lambda = \hat{k}_1/\hat{k}_2$ which is exactly the Dundurs's constant β. This agrees with the result given early by Comninou (1977b).

5. Solutions of the singular integral equations

5.1. Homogeneous anisotropic material

The derived coupled singular equations presented in Section 3, i.e., Eq. (26) are for the problem composed by anisotropic bimaterial. Before discussing the phenomena for bimaterial it would be fruitful to consider the special case of a homogeneous solid, as shown in Eq. (28d). If the crack faces are treated as frictionless, then the solutions for dislocation densities are sought from the following equations

$$\frac{-1}{2\pi}\int_{-c}^{c}\begin{bmatrix} L_{11} & L_{13} \\ L_{13} & L_{33}\end{bmatrix}\begin{bmatrix} b_1 \\ b_3\end{bmatrix}\frac{d\xi}{\xi - x_1} = \begin{bmatrix} -S \\ 0\end{bmatrix} \tag{62}$$

and the results are

$$\begin{bmatrix} b_1 \\ b_3\end{bmatrix} = \frac{2S}{L_{11}L_{33} - L_{13}^2}\begin{bmatrix} L_{33} \\ -L_{13}\end{bmatrix}\frac{x_1}{\sqrt{c^2 - x_1^2}}, \quad |x_1| \leqslant c \tag{63}$$

which are both square root singular at ends of the crack tips and the dislocation densities for isotropic materials may be recovered from above, i.e.,

$$\begin{bmatrix} b_1 \\ b_3\end{bmatrix} = \frac{S}{\mu\kappa}\begin{bmatrix} 1 \\ 0\end{bmatrix}\frac{x_1}{\sqrt{c^2 - x_1^2}}, \quad |x_1| \leqslant c \tag{64}$$

where $\kappa = 3 - 4\nu$ for plain strain and $\kappa = (3 - 4\nu)/(1 + \nu)$ for plain stress. The traction $\mathbf{t}_2$ over the crack faces may be evaluated by the formula

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0\end{bmatrix} + \frac{-1}{2\pi}\int_{-c}^{c}\begin{bmatrix} L_{11}b_1 + L_{13}b_3 \\ L_{12}b_1 + L_{23}b_3 \\ L_{13}b_1 + L_{33}b_3\end{bmatrix}\frac{d\xi}{\xi - x_1} \tag{65}$$

and the result is

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} 0 \\ -\left(N + \frac{(L_{12}L_{33} - L_{23}L_{13})S}{L_{11}L_{33} - L_{13}^2}\right) \\ 0\end{bmatrix} \tag{66}$$

It is seen that for general anisotropic material the remote applying shear load S will induce normal stress on the crack faces. This is due to our assumption that the crack faces have no displacement jump in the direction perpendicular to the crack faces. These induced normal forces may be zero if the anisotropic material has the property

$$L_{12}L_{33} = L_{13}L_{23}. \tag{67}$$

Note that this property is automatically satisfied for isotropic material since $L_{12} = L_{13} = L_{23} = 0$. This condition stated in Eq. (67) is immediately clear by noting that the displacements of the crack faces in x_2-direction for a homogeneous isotropic material under the only remote shear loading S are

$$u_2^+ = u_2^- = (\kappa + 1)Sx_1/(4\mu), |x_1| \leqslant c \tag{68}$$

which immediately shows that the jump is zero across the crack faces. Hence no normal force is induced when only shear load is applied. However when the solid is considered to be in general of anisotropic, the displacement jump in x_2-direction would be (Ting, 1996)

$$u_2^+ - u_2^- = 2\sqrt{c^2 - x_1^2}(\boldsymbol{L}^{-1})_{21}S \tag{69}$$

under the remote shear load only. This jump is usually not zero unless $(\boldsymbol{L}^{-1})_{21} = 0$. The condition $(\mathbf{L}^{-1})_{21} = 0$ is actually equivalent to the condition stated above, i.e., $L_{12}L_{33} = L_{13}L_{23}$ which explains the vanishing of the normal stress over the crack faces for such material. In order to maintain the assumption that $\sigma_{22} \leqslant 0$ over the crack faces, the applied normal stress has to satisfy the condition

$$N \geqslant \frac{(L_{23}L_{13} - L_{12}L_{33})S}{L_{11}L_{22} - L_{13}^2} \tag{70}$$

The stresses on the crack line may also be evaluated as

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0 \end{bmatrix} - \begin{bmatrix} S \\ \frac{S(L_{12}L_{33} - L_{23}L_{13})}{L_{11}L_{33} - L_{13}^2} \\ 0 \end{bmatrix} \left\{ 1 - \frac{x_1 \operatorname{sgn} x_1}{\sqrt{x_1^2 - c^2}} \right\}, \quad |x_1| \geqslant c \tag{71}$$

Next let's suppose the crack faces are not frictionless, then the corresponding equations for the dislocation densities are (28d). The solutions of these equations are

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{\varDelta} \begin{bmatrix} L_{33} + fL_{23} & -(L_{13} + fL_{23}) \\ -(L_{13} + fL_{21}) & L_{11} + fL_{12} \end{bmatrix} \begin{bmatrix} -S + fN \\ fN \end{bmatrix} \frac{2x_1}{\sqrt{c^2 - x_1^2}} \tag{72}$$

where

$$\varDelta = (L_{12}(L_{33} - L_{13}) + L_{23}(L_{11} - L_{13}))f + L_{11}L_{33} - L_{13}^2$$

assuming that $\varDelta \neq 0$. The solution of Eq. (72) reduces to (64) when $f = 0$. If the frictional coefficient f satisfies the following equation:

$$(L_{12}(L_{33} - L_{13}) + L_{23}(L_{11} - L_{13}))f + L_{11}L_{33} - L_{13}^2 = 0 \tag{73}$$

then $\varDelta$ will vanish and this means that the solutions of the dislocation densities b_1 and b_3 will be nonunique for the case when frictional coefficient satisfy Eq. (73). The physical meaning of this situation is not clear and we will assume that $\varDelta \neq 0$ in the following discussions. Substituting (72) into Eq. (65), the stresses acting on the crack faces may be obtained as

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0 \end{bmatrix} + \frac{1}{\varDelta} \begin{bmatrix} (-S + fN)(L_{11}L_{33} - L_{13}^2) - Sf(L_{11}L_{23} - L_{13}L_{21}) \\ (-S + fN)(L_{12}L_{33} - L_{23}L_{13}) + fN(L_{11}L_{23} - L_{12}L_{13}) \\ fN(L_{11}L_{33} - L_{13}^2) - Sf(L_{13}L_{23} - L_{21}L_{33}) \end{bmatrix} \tag{74}$$

It is interesting to observe that the stresses over the crack faces have been redistributed due to the existence of the surface roughness. To see more definitely, first let S = 0 in Eq. (74) and let us consider the solid under one single compressive load. The stresses over the crack faces are $\sigma_{22} = -(L_{11}L_{33} - L_{13}^2)N/\Delta$ and $\sigma_{12} = \sigma_{32} = -f\sigma_{22}$. It is seen that the redistributed normal stress on the crack faces is usually not equal to the applied normal stress N unless either

$$f = 0$$

or

$$L_{11}L_{23} + L_{33}L_{12} = L_{13}(L_{12} + L_{23}) \tag{75}$$

is satisfied. The above two conditions are reached by letting $\Delta = L_{11}L_{33} - L_{13}^2$. The first condition of frictionless surface is easily understood. The satisfaction of the second condition of an anisotropic material allows the material to be able to transmit normal stress across the crack faces without any interference, i.e., $\sigma_{22} = -N$, even if the surfaces have the roughness property. The meaning of the second condition may be explained by the following arguments. Noting again that the displacements jump of the crack faces in x_1- and x_3-directions for homogeneous anisotropic material under only the remote normal loading N are (Ting, 1996)

$$u_1^+ - u_1^- = \sqrt{c^2 - x_1^2}(\boldsymbol{L}^{-1})_{12}N, \quad |x_1| \leqslant c$$

and

$$u_3^+ - u_3^- = \sqrt{c^2 - x_1^2}(\boldsymbol{L}^{-1})_{32}N, \quad |x_1| \leqslant c$$

respectively. The jump in x_1-direction would vanish if $(\boldsymbol{L}^{-1})_{12}$ is zero. The vanish of $(\boldsymbol{L}^{-1})_{12}$ gives rise to the same condition as Eq. (67), i.e., $L_{12}L_{33} = L_{23}L_{13}$. The vanish of the jump in x_3-direction would require $(\boldsymbol{L}^{-1})_{32} = L_{11}L_{23} - L_{12}L_{13} = 0$. Hence if an anisotropic material of a solid has these two properties, i.e., $(\boldsymbol{L}^{-1})_{12} = (\boldsymbol{L}^{-1})_{32} = 0$, then it is clear that the solid with a central crack would undergo no displacement jump across the crack faces when the solid is subjected to a normal compressive load. This explains the reason of $\sigma_{22} = -N$ on the crack faces for such a material when only normal compressive load is applied. Note that the condition specified in Eq. (75) would be automatically satisfied if the material possesses these two properties $(\boldsymbol{L}^{-1})_{12} = (\boldsymbol{L}^{-1})_{32} = 0$. Next by letting $N = 0$ in Eq. (74) the stresses over the crack faces due to pure shear load S only are

$$\sigma_{22} = -(L_{12}L_{33} - L_{23}L_{13})S/\Delta, \quad \sigma_{12} = \sigma_{32} = -f\sigma_{22} \tag{76}$$

It is seen again that the satisfaction of the condition (67) will lead to the vanish of all stresses on the crack faces when the shear load is applied. To maintain the compressive stress $\sigma_{22} \leqslant 0$ over the crack faces, the applied normal and shear stresses have to satisfy

$$N \leqslant \frac{S(L_{23}L_{13} - L_{12}L_{33})}{(L_{11}L_{33} - L_{13}^2) + L_{23}(L_{12} - L_{13})f^2} \tag{77}$$

5.2. Anisotropic bimaterial

Let us now consider the problem of general anisotropic bimaterial. First we observe from Eq. (26b) that if $\tilde{\boldsymbol{W}}$ is identically zero then the corresponding equation will be identical to that for a homogeneous material, i.e., (28d) and in this situation the square root singular behavior at the tips will be preserved. Hence the corresponding dislocation solutions and the stresses developed on the crack faces for general anisotropic bimaterial are still given by (72) and (74), respectively, with the elements of $\boldsymbol{L}$ replaced by the elements of $2\tilde{\boldsymbol{L}}$. It is interesting to observe that for anisotropic bimaterial, as long as $\tilde{\boldsymbol{W}} = \boldsymbol{0}$, the stresses developed on the crack faces are uniformly distributed no mater whether the friction exists or not. Suppose $\tilde{\boldsymbol{W}}$ is not identically zero but, as mentioned before, one of the elements $\tilde{w}_2 = 0$ and the crack faces are frictionless, then the solution of dislocation densities are still equivalent to the homogeneous one (see Eq. (28a)). However, the stresses developed on the crack faces for this case are quite different. The results are

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = -\begin{bmatrix} 0 \\ \left(N + \frac{(\tilde{L}_{12}\tilde{L}_{33} - \tilde{L}_{23}\tilde{L}_{13})S}{\tilde{L}_{11}\tilde{L}_{33} - \tilde{L}_{13}^2} \right) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{(\tilde{w}_3\tilde{L}_{33} + \tilde{w}_1\tilde{L}_{13})S}{\tilde{L}_{11}\tilde{L}_{33} - \tilde{L}_{13}^2} \\ 0 \end{bmatrix} \frac{x_1}{\sqrt{c^2 - x_1^2}} \tag{78}$$

Noting that the normal stress induced by the shear load has two parts. The first part of uniform stresses has been discussed above. The other part will have square root singular at both ends of the crack and that part is related to the material constants $\tilde{w}_1$ and $\tilde{w}_3$. The singular behavior of stresses will disappear if the bimaterial have further properties such that $\tilde{w}_3 = \tilde{w}_1 = 0$ and this is actually the case of $\tilde{\boldsymbol{W}} = \boldsymbol{0}$ as discussed above. From Eq. (78) we see that near the right crack tip if the normal stress developed on the crack faces is compressive, then the normal stress near the left crack tip would undergo tensile stress no matter how large the normal compressive load N is applied unless the applied shear load S is zero. This phenomenon is unrealistic but it does occur for anisotropic bimaterial.

5.3. Monoclinic bimaterial with symmetry plane at $x_3 = 0$

Suppose the monoclinic bimaterial with symmetry plane at $x_3 = 0$ were considered, it is seen that these coupled equations (Eq. (26b)) were decoupled as

$$\frac{-1}{\pi}\int_{-c}^{c} \begin{bmatrix} \tilde{L}_{11} + f\tilde{L}_{12} & 0 \\ f\tilde{L}_{21} & \tilde{L}_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \frac{d\xi}{\xi - x_1} + (-f\tilde{w}_3)\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} -S + fN \\ fN \end{bmatrix}, \quad |x_1| \leqslant c \tag{79}$$

since $\tilde{L}_{13} = \tilde{L}_{23} = 0$ and $\tilde{w}_1 = \tilde{w}_2 = 0$. Two special cases have to be emphasized here, i.e., $f = 0$ and $\tilde{w}_3 = 0$ because for both cases the above equation will produce square root singular behavior at the tips of the crack. The situation when $\tilde{w}_3 = 0$ corresponds to the case of $\tilde{\mathbf{W}} = \mathbf{0}$ discussed above. Considering $f\tilde{w}_3 \neq 0$, then it is observed that the in-plane behavior (first equation of Eq. (79)) is totally decoupled from the anti-plane deformation (second equation of Eq. (79)). However, the anti-plane part is influenced by the in-plane deformation. Solving first equation of Eq. (79) for the unknown $b_1(x_1)$ and then substituting it into the second equation of Eq. (79), the dislocation density $b_3(x_1)$ for the anti-plane deformation can be determined. The result is

$$-\frac{1}{\pi}\int_{-c}^{c} \frac{\tilde{L}_{33}b_3(\xi)}{\xi - x_1}\, d\xi = S - \frac{1}{\pi}\int_{-c}^{c} \frac{\tilde{L}_{11}b_1(\xi)}{\xi - x_1}\, d\xi, \quad |x_1| \leqslant c \tag{80}$$

The right hand side of the above equation represents the induced force in the anti-plane direction for the monoclinic bimaterial when only the in-plane forces, i.e., S and N, are applied to the solid. The induced force in anti-plane direction is due to the assumption of the existence of the frictional surface in that direction. The induced force will vanish if the crack faces are treated as frictionless. It is easy to verify that the right hand side of the above equation is zero when $f = 0$. This is an obvious result because for the monoclinic bimaterial the in-plane deformation is totally decoupled from the anti-plane problem if the crack faces are frictionless.

Before solving Eq. (79) for the in-plane mode response, one should notice that if the order of singularity at the right crack tip is δ with a known direction of the relative movement of the upper and lower crack faces being chosen as positive frictional coefficient, then the order of singularity at the left crack tip would be $1 - \delta$ since the relative movement of the upper and lower crack faces with respect to the left crack tip is just opposite to the right crack tip which accounts for the frictional coefficient should take the minus sign for the left crack tip. This will result in a different order of singularity for the left crack tip. The singular integral equation (79) for $b_1(x_1)$ may be solved analytically by using the formula proposed by Mikhlin (1964). To obtain a unique solution for that equation, the auxiliary condition of single displacements around a closed contour surrounding the whole crack should be enforced as in Eq. $(27)_1$. Once the analytic results for the dislocation density $b_1(x_1)$ is obtained, density for b_3 may be evaluated by Eq. (80). The results are

$$\begin{bmatrix} b_1(x_1) \\ b_3(x_1) \end{bmatrix} = \begin{bmatrix} (S - fN) \\ -(S + fN)f\tilde{L}_{11}/\tilde{L}_{33} \end{bmatrix} \frac{\sin(\delta\pi)}{\tilde{L}_{11} + f\tilde{L}_{12}} \frac{x_1 - (2\delta - 1)c}{(c - x_1)^{\delta}(c + x)^{1-\delta}}, \quad |x_1| \leqslant c \tag{81}$$

with δ being given by Eq. (56). It is noted here that the singular nature for b_3 is due to the frictional shear force developed on the crack faces, which is itself singular for most cases. It is also noted that the problem of frictional interfacial crack under combined shear and compression for isotropic bimaterials has been treated by Qian and Sun (1998). Our derived singular equation for dislocation density b_1, valid here for monoclinic bimaterials, is the same form as their's. Hence, our result of Eq. (81) for b_1 may be reduced to that for isotropic bimaterials. As we mentioned above, the order of sigularities for each crack tip is either δ or $(1 - \delta)$, and the order will be different for each crack tip except the two special cases, either $f = 0$ or $\tilde{w}_3 = 0$ are considered. For example the dislocation densities for the case $\tilde{w}_3 = 0$ are

$$\begin{bmatrix} b_1(x_1) \\ b_3(x_1) \end{bmatrix} = \begin{bmatrix} (S - fN) \\ -(S + fN)f\tilde{L}_{11}/\tilde{L}_{33} \end{bmatrix} \frac{x_1}{\sqrt{c^2 - x_1^2}} \tag{82}$$

The dislocation densities corresponding to frictionless case may be obtained by setting $f = 0$ in the above equation. With known dislocation densities the stress fields on the interface may be determined as

For $|x_1| < c$

$$\begin{aligned} &\sigma_{21}(x_1, 0) = \sigma_{23} = -f\sigma_{22}(x_1, 0) \\ &\sigma_{22}(x_1, 0) = -N - \frac{(S - fN)\tilde{L}_{11}}{\tilde{L}_{11} + f\tilde{L}_{12}} \left(\Lambda \sin(\delta\pi) \frac{x_1 - (2\delta - 1)c}{(c - x_1)^{1-\delta}(c + x_1)^{\delta}} + \frac{\tilde{L}_{12}}{\tilde{L}_{11}} \right) \end{aligned} \tag{83a}$$

For $|x_1| > c$

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{L}_{11}(S - fN) \\ \tilde{L}_{12}(S - fN) \\ f\tilde{L}_{11}(-S - fN) \end{bmatrix} \frac{1}{\tilde{L}_{11} + f\tilde{L}_{12}} \left\{ \frac{x_1 - (2\delta - 1)c}{(x_1 - c)^{\delta}(x_1 + c)^{1-\delta}} - 1 \right\} \tag{83b}$$

It is noted that for the frictionless case the stresses on the interface may be obtained from above by letting $f = 0$ and $\delta = 1/2$ which are

For $|x_1| < c$

$$\begin{aligned} &\sigma_{21}(x_1, 0) = \sigma_{23} = -f\sigma_{22}(x_1, 0) \\ &\sigma_{22}(x_1, 0) = -N - S\left(\Lambda \frac{x_1}{\sqrt{c^2 - x_1^2}} + \frac{\tilde{L}_{12}}{\tilde{L}_{11}} \right) \end{aligned} \tag{84a}$$

For $|x_1| > c$

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0 \end{bmatrix} + \begin{bmatrix} S \\ (\tilde{L}_{12}/\tilde{L}_{11})S \\ 0 \end{bmatrix} \left\{ \frac{1}{\sqrt{x_1^2 - c^2}} - 1 \right\} \tag{84b}$$

It is seen that these results will be reduced to those for isotropic homogeneous materials. It is emphasized again that for the case $\tilde{w}_3 = 0$ the stresses developed on the crack faces for monoclinic bimaterials are uniform. Setting $\tilde{w}_3 = 0$ in Eq. (83), we get

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} S \\ -N \\ 0 \end{bmatrix} + \frac{1}{\tilde{L}_{11} + f\tilde{L}_{12}} \begin{bmatrix} \tilde{L}_{11}(-S + fN) \\ \tilde{L}_{12}(-S + fN) \\ (\tilde{L}_{11} + \tilde{L}_{21})fN \end{bmatrix} \tag{85}$$

which may also be obtained from Eq. (74) by substituting appropriate material constants in that equation. As to the case when $f = 0$, the stresses over crack faces are

$$\begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} = -\begin{bmatrix} 0 \\ N + \frac{S}{\tilde{L}_{11}}\left(\frac{\tilde{w}_3 x_1}{\sqrt{c^2 - x_1^2}} + \tilde{L}_{12}\right) \\ 0 \end{bmatrix}, \quad |x_1| \leqslant c \tag{86}$$

which also shows that the normal stress near one of the crack tip may be in tension unless the shear load S or $\tilde{w}_3$ vanishes. One final remark is that the once the stress fields have been determined, the stress intensity factors at the crack tips can be extracted directly. For instance, the stress intensity factors at right crack tip defined by

$$(K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}) = \lim_{x_1 \to c} (2\pi(x_1 - c))^{\delta} (\sigma_{21}, \sigma_{22}, \sigma_{23})) \tag{87}$$

may be easily evaluated from above given stresses fields.

6. Conclusions

A frictionally sliding interface crack embedded in an infinite anisotropic bimaterial subjected to remote shear and normal compression load is analyzed in this paper. The frictional resistance is assumed to be uniform over the sliding zone. A set of singular integral equations is formulated. The singularities at the sliding crack tip are analyzed and the dependence on the material constants is noted. Some interesting phenomena are observed for anisotropic bimaterial from the investigations of the singular integral equations. For monoclinic bimaterial, results are given in analytic form. The obtained stresses field on the interface may be reduced to those appearing in the literature.

Appendix A

Starting from the Lekhnitskii's formalism, the stress singularities investigated in Section 4 may also be developed. It is known that for monoclinic homogeneous materials with symmetric plane at $x_3 = 0$ the matrices **S**, **H** and **L** are (Lixin and Ting, 1996)

$$\mathbf{S} = \frac{s^2}{g'} \begin{bmatrix} d' & -b' & 0 \\ e' & -d' & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.1a}$$

$$\mathbf{H} = s'_{11}(1 - s^2) \begin{bmatrix} b' & d' & 0 \\ d' & e' & 0 \\ 0 & 0 & \chi \end{bmatrix}, \quad \left(\chi = \frac{1}{\mu s'_{11}(1 - s^2)}\right) \tag{A.1b}$$

$$\mathbf{L} = \frac{s^2}{s'_{11}g'^2}\begin{bmatrix} e' & -d' & 0 \\ -d' & b' & 0 \\ 0 & 0 & \psi \end{bmatrix}, \quad (\psi = \mu s'_{11} g'^2 s^{-2}) \tag{A.1c}$$

and

$$\mathbf{SL}^{-1} = w\mathbf{J} \tag{A.2a}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad w = g's'_{11} \tag{A.2b}$$

Constants a' to g' are related to p_α as follows

$$p_1 + p_2 = a' + \mathrm{i}b' \ (b' > 0), \quad p_1 p_2 = c' + \mathrm{i}d' \tag{A.3a}$$

$$e' = \mathrm{Im}\{p_1 p_2(\bar{p}_1 + \bar{p}_2)\} > 0, \quad g' = s'_{12}/s'_{11} - c' > 0, \quad 1 > s = \frac{g'}{\sqrt{b'e' - d'^2}} > 0 \tag{A.3b}$$

Therefore matrices $\hat{\mathbf{W}}$ and $\hat{\mathbf{D}}$ defined in Eqs. (45) and (46) are

$$\hat{\mathbf{W}} = \begin{bmatrix} 0 & \hat{w}_3 & 0 \\ -\hat{w}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (g'^{(1)}s'^{(1)}_{11} - g'^{(2)}s'^{(2)}_{11})\mathbf{J} = (w^{(1)} - w^{(2)})\mathbf{J} = -\hat{w}_3\mathbf{J} \tag{A.4a}$$

$$\hat{\mathbf{D}} = \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} & 0 \\ \hat{D}_{12} & \hat{D}_{22} & 0 \\ 0 & 0 & \hat{D}_{33} \end{bmatrix} = \begin{bmatrix} b'^{(1)}s'^{(1)}_{11} + b'^{(2)}s'^{(2)}_{11} & d'^{(1)}s'^{(1)}_{11} + d'^{(2)}s'^{(2)}_{11} & 0 \\ d'^{(1)}s'^{(1)}_{11} + d'^{(2)}s'^{(2)}_{11} & e'^{(1)}s'^{(1)}_{11} + e'^{(2)}s'^{(2)}_{11} & 0 \\ 0 & 0 & \eta_1 + \eta_2 \end{bmatrix} \tag{A.4b}$$

where the reduced elastic compliance s'_{mn} are related to the elastic compliance s_{mn} by

$$s'_{mn} = s_{mn} - \frac{s_{m3}s_{3n}}{s_{33}}, \quad (m, n = 1, 2, 3) \tag{A.5}$$

Hence the characteristic equation in terms of reduced elastic compliance is

$$(\mathrm{e}^{2\mathrm{i}\pi\delta} - 1)^3[-f\hat{w}_3\hat{D}_{33} + \mathrm{i}\lambda(f\hat{D}_{12} - \hat{D}_{22})\hat{D}_{33}][\mathrm{i}\lambda(\hat{w}_3^2 + \hat{D}_{12}^2 - \hat{D}_{11}\hat{D}_{22})] = 0 \tag{A.6}$$

which implies

$$\cot\delta\pi = -f\Lambda \quad \text{and} \quad \cot\delta\pi = 0 \tag{A.7}$$

where

$$\Lambda = \frac{w^{(1)} - w^{(2)}}{s'^{(1)}_{11}(e'^{(1)} - fd'^{(1)}) + s'^{(2)}_{11}(e'^{(2)} - fd'^{(2)})} \tag{A.8}$$

It can be shown that this Λ is the same as

$$\Lambda = \tilde{w}_3/(\tilde{L}_{11} + f\tilde{L}_{12}) \tag{A.9}$$

defined before.

References

Comninou, M., 1977a. Interface crack with friction in the contact zone. ASME J. Appl. Mech. 44, 780–781.
Comninou, M., 1977b. The interface crack. ASME J. Appl. Mech. 44, 631–636.
Comninou, M., Dundurs, J., 1983. Spreading of slip from a region of low friction. Acta Mech. 47, 65–71.
Gorbatikh, L., Nuller, B., Kachanov, M., 2001. Sliding on cracks with non-uniform frictional characteristics. Int. J. Solids Struct. 38, 7501–7524.
Hoek, E., Se, M., Bieniawski, Z.T., 1984. Brittle fracture propagation in rock under compression. Int. J. Fract. 26, 276–294.
Horii, H., Nemat-Nasser, S., 1982. Instability of a half-space with frictional materials. ZAMP J. Appl. Math. Phys. 33, 1–16.
Lauterbach, B., Gross, D., 1998. Crack growth in brittle solids under compression. Mech. Mater. 29, 81–92.
Lixin, W., Ting, T.C.T., 1996. Explicit expressions of the Barnett–Lothe tensors for anisotropic materials. J. Elasticity 36, 67–83.
Mikhlin, S.G., 1964. Integral Equations. Macmillan, New York.
Poonsawat, P., Wijeyewickrema, A.C., Karasudhi, P., 2001. Singular stress fields of angle-ply and monoclinic bimaterial. Int J. Solids Struct. 38, 91–113.
Qian, W., Sun, C.T., 1998. A frictional crack under combined shear and compression. Compos. Sci. Tech. 58, 1753–1761.
Ting, T.C.T., 1996. Anisotropic Elasticity: Theory and Applications. Oxford University Press, New York.